MATH 5061 Solution to Problem Set 2[1]

1. Prove that the antipodal map $A(p) = -p$ induces an isometry on $\mathbb{S}^n$. Use this to introduce a Riemannian metric on $\mathbb{RP}^n$ such that the projection map $\pi : \mathbb{S}^n \to \mathbb{RP}^n$ is a local isometry.

   Solution:

   Firstly, note that antipodal map $A(p) = -p$ will give an isometry on $\mathbb{R}^{n+1}$. That is, let g be the metric on $\mathbb{R}^{n+1}$, then

$$
\begin{aligned}
(A^*g)_p(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j}) &= g_{-p}(dA_p(\frac{\partial}{\partial x_i}), dA_p(\frac{\partial}{\partial x_j})) \\
&= g_{-p}(-\frac{\partial}{\partial x_i}, -\frac{\partial}{\partial x_j}) = \delta_{ij} \\
&= g_{-p}(\frac{\partial}{\partial x_i}, \frac{\partial}{\partial x_j})
\end{aligned}
$$

   So $A^*g = g$. Hence $A^*(g|_{\mathbb{S}^n}) = g|_{\mathbb{S}^n}$, A will induce an isometry on $\mathbb{S}^n$.

   Now we have the nature definition of metric $\tilde{g}$ on $\mathbb{RP}^n$ defined by

$$
\tilde{g}_q(v, w) = g_p|_{\mathbb{S}^n}(v_0, w_0)
$$

   where $q \in \mathbb{RP}^n, p \in \pi^{-1}(q)$, $v_0 \in d\pi_p^{-1}(v), w_0 \in d\pi_p^{-1}(w)$. Note that v_0, w_0 is uniquely determined by v, w since $d\pi_p$ is a isomorphism. We need to verity $\tilde{g}$ is well-defined.

   If p' is another p such that $\pi(p') = q$, then $p' = -p = A(p)$. Hence $g_p|_{\mathbb{S}^n}(v_0, w_0) = g_{A(p)}|_{\mathbb{S}^n}(dA_p(v_0), dA_p(w_0))$. Note that $d\pi_{A(p)} \circ dA_p = \pi_p$ by $\pi \circ A = \pi$, so p' will give the same definition with p.

   By the construction above, we can find π is indeed a local isometry since locally they are diffeomorphism and their metric is related by π.

2. Show that the isometry group of $\mathbb{S}^n$, with the induced metric from $\mathbb{R}^{n+1}$, is the orthogonal group $O(n+1)$.

   Solution:

   Let $\mathcal{F} := \{F : \mathbb{S}^n \to \mathbb{S}^n | F \text{ is an isometry }\}$. Then we know $O(n+1) \subset \mathcal{F}$ since the orthogonal transformation will keep the metric of $\mathbb{R}^{n+1}$ and hence keep the metric on $\mathbb{S}^n$.

   We will show that $O(n+1) = \mathcal{F}$.

   Let $\varphi \in \mathcal{F}$ be an isometry of $\mathbb{S}^n$. Then we construct a new map $\psi : \mathbb{R}^{n+1} \backslash \{0\} \to \mathbb{R}^{n+1} \backslash \{0\}$ in the following ways

$$
\psi(x) = |x| \, \varphi(\frac{x}{|x|}), x \in \mathbb{R}^{n+1} \backslash \{0\}.
$$

   One can verify this is a diffeomorphism. Moreover, we can calculate the differential map at x with direction v as following, (e.g. calculating $\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\psi(c(t))$

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with $c(0) = x, c'(0) = v$)

$$\begin{aligned}
d\psi_x(v) &= \varphi\left(\frac{x}{|x|}\right)\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\,|x+tv| + |x|\,\frac{\mathrm{d}}{\mathrm{d}t}|_{t=0}\varphi\left(\frac{x+tv}{|x+tv|}\right)\\
&= \frac{\langle x,v\rangle}{|x|^2}\varphi\left(\frac{x}{|x|}\right) + |x|\,d\varphi_{\frac{x}{|x|}}\left(\frac{v}{|x|} - \frac{\langle x,v\rangle\,x}{|x|^3}\right)
\end{aligned}$$

where $\langle\cdot,\cdot\rangle$ is the inner product on $\mathbb{R}^{n+1}$ (Or the standard metric on Euclidean space)

Use the fact φ is an isometry, i.e. $\langle d\phi_p(v), d\phi_p(w)\rangle = \langle v, w\rangle$, and then fact $\varphi(\frac{x}{|x|})\perp \operatorname{Im}\left(d\phi_{\frac{x}{|x|}}\right), \left|\varphi(\frac{x}{|x|})\right| = 1$, we find

$$\begin{aligned}
\langle d\psi_x(v), d\psi_x(w)\rangle &= \frac{\langle x,v\rangle\langle x,w\rangle}{|x|^4} + |x|^2\left\langle \frac{v}{|x|} - \frac{\langle x,v\rangle\,x}{|x|^3}, \frac{w}{|x|} - \frac{\langle x,w\rangle\,x}{|x|^3}\right\rangle\\
&= \langle v,w\rangle\,.
\end{aligned}$$

So we get $\psi : \mathbb{R}^{n+1}\backslash\{0\} \to \mathbb{R}^{n+1}\backslash\{0\}$ is an isometry. Now we can use the properties of Euclidean space to show ψ is indeed a linear map.

Since ψ is an isometry, it keeps the distance of different points. That is, if $p, q \in \mathbb{R}^{n+1}\backslash\{0\}$, such that the line segment pq doesn't contain 0, then $|\psi(p) - \psi(q)| = |p - q|$. If the line segment pq contains 0, since ψ is continuous, we still have the same result since we can choose $q_i \to q$ such that pq_i does not contain 0 and take limit in $|\psi(p) - \psi(q_i)| = |p - q_i|$.

Again, by the definition of ψ, we know ψ keeps the length of points. That is

$$|\psi(p)| = |p|\left|\varphi(\frac{p}{|p|})\right| = |p|\,.$$

Hence ψ keeps the inner product by the following

$$\begin{aligned}
\langle\psi(p),\psi(q)\rangle = \frac{1}{2}\left(|\psi(p)|^2 + |\psi(q)|^2 - |\psi(p)-\psi(q)|^2\right) &= \frac{1}{2}\left(|p|^2 + |q|^2 - |p-q|^2\right)\\
&= \langle p,q\rangle
\end{aligned}$$

for any $p, q \in \mathbb{R}^{n+1}\backslash\{0\}$.

So for any $a, b \in \mathbb{R}, p, q, r \in \mathbb{R}^{n+1}\backslash\{0\}$, we have

$$\langle\psi(ap+bq) - a\psi(p) - b\psi(q), \psi(r)\rangle = \langle ap+bq, r\rangle - a\langle p,r\rangle - b\langle q,r\rangle = 0$$

Note that $\psi(r)$ can take any vectors in $\mathbb{S}^n$, by choose $\psi(r) = e_1, \cdots, e_{n+1}$ to be the basis of $\mathbb{R}^{n+1}$, we actually know

$$\psi(ap+bq) = a\psi(p) + b\psi(q).$$

Hence if we define $\psi(0) = 0$, we actually get $\psi : \mathbb{R}^{n+1} \to \mathbb{R}^{n+1}$ is a linear map. It is an orthogonal map since ψ also keeps the length of any line segments of $\mathbb{R}^{n+1}$.

So as a restriction of ψ, the map φ is an orthogonal transformation on $\mathbb{S}^n$.

3. For any smooth curve $c : I \to M$ and $t_0, t \in I$, we denote the parallel transport map as $P = P_{c,t_0,t} : T_{c(t_0)}M \to T_{c(t)}M$ along c from $c(t_0)$ to $c(t)$.

   (a) Show that P is a linear isometry. Moreover, if M is oriented, then P is also orientation-preserving.

   (b) Let X, Y be vector fields on M, $p \in M$. Suppose $c : I \to M$ is an integral curve of X with $c(t_0) = p$. Prove that

$$(\nabla_X Y)(p) = \left.\frac{d}{dt}\right|_{t=t_0} P^{-1}_{c,t_0,t}(Y(c(t))).$$

   Solution:

(a). Let write a new curve $\tilde{c}(s) = c(t + t_0 - s)$ from $c(t)$ to $c(t_0)$ for $s \in [t_0, t]$. So we can define the new map $\tilde{P} = P_{\tilde{c},t,t_0} : T_{c(t)}M \to T_{c(t_0)}M$.

Note $P, \tilde{P}$ are all homomorphism since for constant a, b, we always have $\nabla_X(aY + bZ) = a\nabla_X Y + b\nabla_X Z$.

Let's show $\tilde{P} \circ P = \mathrm{Id}_{T_{c(t_0)}M}$. This is because, for any $V(c(t))$, the parallel transportation of $V \in T_{c(t_0)}M$ along c, we consider the vector fields $V(c(s)) = V(\tilde{c}(t + t_0 - s))$, we have

$$\nabla_{\tilde{c}'(s)}V = \nabla_{-c'(s)}V = 0$$

Hence $V(\tilde{c}(t + t_0 - s))$ is a parallel transport from $V(c(t))$ along $\tilde{c}$. Hence $\tilde{P}(V(c(t))) = V(c(t_0))$. That's $\tilde{P} \circ P(V(c(t_0))) = V(c(t_0))$.

Similarly, we know $P \circ \tilde{P} = \mathrm{Id}_{T_{c(t)}}M$. Hence P is an isomorphism.

For the linear isometry, Let V, W be two vectors fields that are all paralleled along c. Since the metric is compatible with connection, we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}g(V(s), W(s)) &= g(\nabla_{c'(s)}V(s), W(s)) + g(V(s), \nabla_{c'(s)}W(0)) \\ &= g(0, W(s)) + g(V(s), 0)) = 0\end{aligned}$$

Integrate s from t_0 to t, we have $g(V(t), W(t)) = g(V(t_0), W(t_0))$.

If M is orientable, we consider the $P_s = P_{c,t_0,s}$ for any $s \in [t_0, t]$. Let's choose an orientable basis $e_1, \cdots e_n \in T_{c(t_0)}M$ and let $e_i(s) = P_s(e_i)$, the parallel transport of e_i along c.

Let's consider the function $f(s) : [t_0, t] \to \{-1, 1\}$ where $f(s) = 1$ if and only if P_s is orientation-preserving.

Clearly $f(s)$ is continuous since in any oriented local coordinate chart $x_1, \cdots, x_n$, we write $e_i = \sum_{j=1}^n a_{ij} \frac{\partial}{\partial x_j}$, then orientation of $e_i(s)$ is determined by the sign of $\det(a_{ij}(s))$, which is continuous with respect to s.

Since $f(t_0) = 1$, we get $f(s) = 1$ for all $s \in [t_0, t]$. So P is orientation preserving.

(b).

As before, we choose $e_1, \cdots, e_n$ as the basis of $T_{c(t_0)}M$, and let $e_i(c(t))$ be the parallel transformation along $c(t)$ from the vectors e_i. Since $e_i(c(t))$ is the basis of $T_{c(t)}M$ by the isomorphism of P, we can write $Y(c(t)) = a_i(t)e_i(c(t))$. Hence

$$\begin{aligned}\nabla_X Y(p) &= \sum_{i=1}^n \nabla_{c'(0)}(a_i(t)e_i(c(t)))|_{t=t_0} = \sum_{i=1}^n c'(0)(a_i(t))e_i(p) + a_i(0)\nabla_{c'(0)}e_i(p) \\ &= \sum_{i=0}^n a_i'(0)e_i(p)\end{aligned}$$

Here $c'(0)(a_i(t))$ means the vector $c'(0)$ acting on the function $a_i(t)$.

On the other hand, use the fact that $P^{-1}_{c,t_0,t}$ is a linear map, we have

$$\begin{aligned}
\left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} P^{-1}_{c,t_0,t}(Y(c(t))) &= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \sum_{i=1}^{n} a_i(t) P^{-1}_{c,t_0,t}(e_i(c(t))) \\
&= \left.\frac{\mathrm{d}}{\mathrm{d}t}\right|_{t=t_0} \sum_{i=1}^{n} a_i(t) e_i(c(t_0)) \\
&= \sum_{i=0}^{n} a_i'(0) e_i(p)
\end{aligned}$$

Hence $(\nabla_X Y)(p) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=t_0} P^{-1}_{c,t_0,t}(Y(c(t)))$.

4. Prove the *second Bianchi identity*: for any vector fields $X, Y, Z, W, T \in \Gamma(TM)$,

$$(\nabla_X R)(Y, Z, W, T) + (\nabla_Y R)(Z, X, W, T) + (\nabla_Z R)(X, Y, W, T) = 0.$$

Solution:

We use the normal coordinate to compute the second Bianchi Identity. Choose $p \in M$ with the normal coordinate $e_1, \cdots, e_n$ at p. So we have $\nabla_{e_i} e_j = 0$ at p for any $1 \leq i, j \leq n$ and hence $[e_i, e_j] = 0$ at p.

So at p, the coderivative of Riemann curvature tensor can be written as

$$\begin{aligned}
(\nabla_{e_i} R)(e_j, e_k, e_l, e_m) &= \frac{\partial}{\partial x_i} R(e_j, e_k, e_l, e_m) \\
&= -\left\langle \nabla_{e_i} \nabla_{e_j} \nabla_{e_k} e_l, e_m \right\rangle + \left\langle \nabla_{e_i} \nabla_{e_k} \nabla_{e_j} e_\lambda e_m \right\rangle
\end{aligned}$$

So

$$\begin{aligned}
&(\nabla_{e_i} R)(e_j, e_k, e_l, e_m) + (\nabla_{e_j} R)(e_k, e_i, e_l, e_m) + (\nabla_{e_k} R)(e_i, e_j, e_l, e_m) \\
= &-\boxed{\left\langle \nabla_{e_i} \nabla_{e_j} \nabla_{e_k} e_l, e_m \right\rangle} + \underline{\left\langle \nabla_{e_i} \nabla_{e_k} \nabla_{e_j} e_l, e_m \right\rangle} \\
&- \underline{\underline{\left\langle \nabla_{e_j} \nabla_{e_k} \nabla_{e_i} e_l, e_m \right\rangle}} + \boxed{\left\langle \nabla_{e_j} \nabla_{e_i} \nabla_{e_k} e_l, e_m \right\rangle} \\
&- \underline{\left\langle \nabla_{e_k} \nabla_{e_i} \nabla_{e_j} e_l, e_m \right\rangle} + \underline{\underline{\left\langle \nabla_{e_k} \nabla_{e_j} \nabla_{e_i} e_l, e_m \right\rangle}} \\
= &R(e_i, e_j, \nabla_{e_k} e_l, e_m) + R(e_j, e_k, \nabla_{e_i} e_l, e_m) + R(e_k, e_i, \nabla_{e_j} e_l, e_m) \\
= &0 \quad (\nabla_{e_i} e_j = 0 \text{ for } 1 \leq i, j \leq n \text{ and } R \text{ is a tensor. })
\end{aligned}$$

Since the coderivative of R is still a tensor, then by the linearity of R, we have

$$(\nabla_X R)(Y, Z, W, T) + (\nabla_Y R)(Z, X, W, T) + (\nabla_Z R)(X, Y, W, T) = 0$$